Ch11-Square-Root Adaptive Filters

- One of the problems encountered in applying the RLS algorithm is that of numerical instability, which can arise because of the way in which the Riccati difference equation is formulated.
- The solution to the square-Root Kalman filtering problem sets the stage for deriving the corresponding variants of the algorithm in light of the one-to-one correspondences that exits between the Kalman variables and the RLS variables, established in the previous chapter.

$$\mathbf{K}(n) = \mathbf{K}^{\frac{1}{2}}(n)\mathbf{K}^{\frac{H}{2}}(n)$$

- Where $\mathbf{K}^{H/2}(n)$ is Hermitian transpose of $\mathbf{K}^{1/2}(n)$.
- $\mathbf{K}^{1/2}(n)$ is a lower triangular matrix and is defined as the square root of **K**(n).
- Unlike the situation that may exist with the covariance Kalman filter, the nonnegative definite character of **K**(n) as a correlation matrix preserved by virtue the fact that the product of any square matrix and its Hermitian transpose is always a nonnegative definite matrix.

$$\mathbf{x}(n+1) = \lambda^{-1/2}\mathbf{x}(n)$$

$$y(n) = \mathbf{u}^{H}(n)\mathbf{x}(n) + v(n),$$

- where $\mathbf{x}(n)$ the state vector, the row vector $\mathbf{u}^H(n)$ is the measurement matrix, the scalar $y(n)$ is an observation or reference signal, the scalar $v(n)$ is a white-noise process of zero mean and unit variance, and the positive real scalar λ is a constant of the model.

Matrix Factorization lemma

- Given any two N-by-M matrices A and B with dimension N < M, the matrix factorization lemma states that:

$$\mathbf{A}\mathbf{A}^H = \mathbf{B}\mathbf{A}^H$$

- if, and only if, there exists a unitary matrix $\mathbf{\Theta}$ such that $\mathbf{B} = \mathbf{A}\Theta$

$$\mathbf{B}\mathbf{B}^H = \mathbf{A}\Theta\Theta^H\mathbf{A}^H$$

$$\Theta\Theta^H = \mathbf{I}$$

- We may prove the converse implication of the matrix factorization lemma by invoking the singular-value decomposition theorem, according to which the matrix **A** may factored as

$$\mathbf{A} = \mathbf{U}_A \boldsymbol{\Sigma}_A \mathbf{V}_A^H$$

- where $\mathbf{U}_A$ and $\mathbf{V}_A$ are N-by-N and M-by-M unitary matrixes, respectively and $\boldsymbol{\Sigma}_A$ is an N-by-M matrix defined by the singular values of matrix **A**. Similarly, the second matrix **B** may be factored as

$$\mathbf{B} = \mathbf{U}_B \boldsymbol{\Sigma}_B \mathbf{V}_B^H$$

- The identity $\mathbf{A}\mathbf{A}^H = \mathbf{B}\mathbf{B}^H$ implies that we have

$$\mathbf{U}_A = \mathbf{U}_B$$

$$\boldsymbol{\Sigma}_A = \boldsymbol{\Sigma}_B$$

$$\Theta = \mathbf{V}_A \mathbf{V}_B^H$$

Square-Root Covariance filter

$$\mathbf{K}(n) = \lambda^{-1}\mathbf{K}(n-1) - \lambda^{-1}\mathbf{K}(n-1)\mathbf{u}(n)r^{-1}(n)\mathbf{u}^H(n)\mathbf{K}(n-1)$$

$$r(n) = 1 + \mathbf{u}^H(n)\mathbf{K}(n-1)\mathbf{u}(n)$$

- The following four distinct matrix terms constitute the right-hand side of the above Riccati equation

- Scalar: $\mathbf{u}^H(n)\mathbf{K}(n-1)\mathbf{u}(n)+1$
- 1-by-M vector: $\lambda^{-\frac{1}{2}}\mathbf{u}^H(n)\mathbf{K}(n-1)$
- M-by-1 vector: $\lambda^{-\frac{1}{2}}\mathbf{K}(n-1)\mathbf{u}(n)$
- M-by-M matrix: $\lambda^{-1}\mathbf{K}(n-1)$
- we may arrange these four terms in the form of a block matrix that contains the complete information on **K**(n).

$$\mathbf{H}(n)=\begin{bmatrix}\mathbf{u}^H(n)\mathbf{K}(n-1)\mathbf{u}(n)+1 & \lambda^{-\frac{1}{2}}\mathbf{u}^H(n)\mathbf{K}(n-1)\\ \lambda^{-\frac{1}{2}}\mathbf{K}(n-1)\mathbf{u}(n) & \lambda^{-1}\mathbf{K}(n-1)\end{bmatrix}.$$

- Expressing the correlation matrix **K**(n-1) in its factored form yields

$$\mathbf{K}(n-1)=\mathbf{K}^{\frac{1}{2}}(n-1)\mathbf{K}^{\frac{H}{2}}(n-1)$$

- Using Cholesky factorization

$$\mathbf{H}(n)=\underbrace{\begin{bmatrix}1 & \mathbf{u}^H(n)\mathbf{K}^{1/2}(n-1)\\ \mathbf{0} & \lambda^{-\frac{1}{2}}\mathbf{K}^{1/2}(n-1)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1 & \mathbf{0}^T\\ \mathbf{K}^{H/2}(n-1)\mathbf{u}(n) & \lambda^{-\frac{1}{2}}\mathbf{K}^{H/2}(n-1)\end{bmatrix}}_{A^H}.$$

$$\underbrace{\begin{bmatrix}1 & \mathbf{u}^H(n)\mathbf{K}^{1/2}(n-1)\\ \mathbf{0} & \lambda^{-\frac{1}{2}}\mathbf{K}^{1/2}(n-1)\end{bmatrix}}_{\mathbf{A}}\mathbf{\Theta}(n)=\underbrace{\begin{bmatrix}b_{11}(n) & \mathbf{0}^T\\ b_{21}(n) & b_{22}(n)\end{bmatrix}}_{\mathbf{B}},$$

- where $\mathbf{\Theta}(n)$ is a unitary rotation and the scalar $b_{11}(n)$, the vector $\mathbf{b}_{21}(n)$, and the matrix $\mathbf{B}_{22}(n)$ denote the nonzero block elements of matrix **B**.
- we may distinguish two arrays of numbers:
- A prearray, **A**, which is operated on by a unitary rotation.
- A postarray, **B**, which is characterized by a block zero entry resulting from the action of the unitary rotation. The postarray therefore has a "triangular" structure in a block sense.